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LETTERS TO THE EDITOR

FUNDAMENTAL FREQUENCY OF A VIBRATING PRETENSIONLESS STRING

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1. INTRODUCTION

Recently, Bolwell [1] introduced a variant of the partial differential equation for a flexible string in which non-linearity is included by simply replacing the constant tension in the linear equation by its Hooke's law expression to take account of finite stretching. When fourth and higher powers were neglected, this gave a non-linear equation for $y(x, t)$, the transverse deflection in a plane:

$$\partial^2 y/\partial t^2 = [c_1^2 + \tfrac{1}{2}c_2^2(\partial y/\partial x)^2]\, \partial^2 y/\partial x^2. \tag{1}$$

Here (cf. [1]), the parameter $c_1 = (T/\rho A)^{1/2}$, which would be the speed of transverse waves along the string in the linear regime (linear transverse wave equation for string tension T), and the parameter $c_2 = (Y/\rho)^{1/2}$, which would be the speed of longitudinal waves along the string in the linear regime (linear longitudinal wave equation). Neither parameter has this significance for the present non-linear wave equation (1). In the above, T is the string pre-stressed tension, ρ is its volume density, A is its cross-sectional area and Y is the Young's modulus. The main interest in reference [1] was in the plucked string and an impact problem.

The consequent increase in tension due to stretching in [1] is in direct contrast to the standard linear D'Alembert string equation, where the tension is assumed unchanged by the small deflections:

$$\partial^2 y/\partial t^2 = c_1^2\, \partial^2 y/\partial x^2. \tag{2}$$

This involves only one speed parameter, in contrast to the two independent speed-like parameters in equation (1).

For model (1), it is evident both on physical grounds, due to tension increasing with amplitude, and mathematical grounds, due to the + sign on the right side, that the fundamental frequency increases as amplitude of vibration increases. By contrast, for the linear D'Alembert string (2), the fundamental frequency remains independent of the (small) amplitude.

2. STRING WITH ZERO INITIAL TENSION

The case of a string, model (1), for which the pre-stressed tension T is zero, i.e., $c_1 = 0$, was discussed to a certain extent in reference [1]. This case is of particular interest *vis-à-vis* the linear D'Alembert wave equation (2), because waves are not possible for this latter case when the tension remains zero. For model (1) of reference [1] on the other hand, on the physical grounds of the increase in tension

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to a positive value when the transverse displacement is non-zero, and due to the mathematical form of equation (1) in general with its two independent speeds so that $T = 0$, i.e., $c_1 = 0$ can be set, the vibrating pretensionless string *is* possible. Then from (1), such a modest string satisfies the partial differential equation

$$\partial^2 y/\partial t^2 = \sigma(\partial y/\partial x)^2\, \partial^2 y/\partial x^2, \tag{3}$$

where

$$\sigma = \tfrac{1}{2}c_2^2; \qquad c_2 = (Y/\rho)^{1/2}. \tag{4}$$

The fundamental frequency of this string (3) will now be studied using various assumed forms together with the harmonic balance method [2, chap. 4] and exact solutions. The boundary conditions are those of fixed ends at $x = 0$ and $x = \mathrm{L}$, viz. $y(x = 0, t) = 0 = y(x = L, t)$.

2.1. *Assumed linear spatial mode*

The assumption is first made in equation (3) that for the fundamental mode

$$y(x, t) = \sin(\pi x/L)T(t). \tag{5}$$

Then a Galerkin-type procedure gives the ordinary nonlinear temporal differential equation satisfied by $T(t)$:

$$\ddot{T} = -\tfrac{1}{4}(\pi^4/L^4)\sigma T^3. \tag{6}$$

2.1.1. *Exact solution of the temporal equation*

Equation (6) is just the standard cubic oscillator equation with given coefficient, whose exact solution for frequency is known [3, p. 4], being expressible in terms of Gamma functions. For scaled frequency

$$\Omega \equiv L^2\omega/c_2, \tag{7}$$

it leads here to the value

$$\Omega_{ex}^T = 2{\cdot}956292\, Y_M, \tag{8}$$

where Y_M is the maximum value of y, here by equation (5) equal to the amplitude T_M of equation (6).

2.1.2. *Harmonic balance solution of the temporal equation*

For the familiar cubic oscillator equation (6), first order harmonic balance ([2, sect. 4.3.1]) using

$$T(t) = a\cos(\omega t) \tag{9}$$

gives the approximate expression for the radian frequency squared:

$$\omega^2 \approx \frac{3}{16}\frac{\pi^4\sigma}{L^4}a^2. \tag{10}$$

Thus, for the overall assumed form

$$y = a\sin(\pi x/L)\cos(\omega t), \tag{11}$$

the first order harmonic balance approximate radian frequency for equation (3) is

$$\omega_1 = (\tfrac{1}{4}\sqrt{3/2}\,\pi^2)(c_2/L^2)T_M \tag{12}$$

where T_M is the maximum amplitude, i.e., $Y_M = T_M = a$ here. Hence

$$\Omega_1 = 3{\cdot}0219\ Y_M. \tag{13}$$

Since the approximation (13) is already within about 2% of the exact value (8) for the temporal equation (6), there is little need to proceed to the even more accurate second-order harmonic balance solution (cf. [2, sect. 4.5.1]) here.

2.2. *Assumed harmonic time dependence*

The assumption in equation (3), that

$$y(x, t) = U(x)\cos\omega t, \tag{14}$$

together with first-order harmonic balance, leads to the ordinary non-linear spatial differential equation for $U(x)$ (where a prime denotes differentiation with respect to x):

$$U''(U')^2 = -[4/(3\sigma)]\omega^2 U, \tag{15}$$

to be solved for the fixed string boundary conditions $U(0) = 0 = U(L)$.

2.2.1. *Exact solution of the spatial equation*

The solution to the spatial equation (15) may be found exactly. A first integral is

$$(U')^4 = [8\omega^2/(3\sigma)][U_M^2 - U^2], \tag{16}$$

where $U_M = U(x = L/2) = Y_M$ is the maximum value of U in equation (15) and hence of y in equation (14). A further integration yields

$$\omega = \sqrt{3}(c_2/L^2)\beta^2 U_M \tag{17}$$

where

$$\beta = \int_0^1 \frac{\mathrm{d}V}{(1 - V^2)^{1/4}} = \int_0^{\pi/2} \sqrt{\cos\theta}\,\mathrm{d}\theta = \tfrac{1}{2}B(\tfrac{1}{2}, \tfrac{3}{4}) = 2\sqrt{2}\pi\sqrt{\pi}/[\Gamma(\tfrac{1}{4})]^2$$

$$= 1{\cdot}1981402 \tag{18}$$

where B is the Beta function [4, p. 258] and Γ is the Gamma function [4, pp. 255–256]. Thus

$$\Omega_{ex}^S = 2{\cdot}486428\ Y_M. \tag{19}$$

The mode shape $U(x)$ itself is given implicitly by

$$\sqrt[4]{8/(3\sigma)}\,\sqrt{\omega/U_M}\,x = \tfrac{1}{2}B_{(U/U_M)^2}(\tfrac{1}{2}, \tfrac{3}{4}) \tag{20}$$

where the function on the right side is the incomplete Beta function [4, p. 263].

It is worthwhile investigating the solution $U(x)$ to equation (15) a little more closely. At $x = 0$, $U = 0$, so by equation (16)

$$U'(x = 0) = \sqrt[4]{8/(3\sigma)}\,\sqrt{\omega}\,\sqrt{U_M}. \tag{21}$$

Since this is non-zero, by equation (15) $U''(x = 0) = 0$.

By using equations (17) and (21), the following relations involving $U'_0 \equiv U'(x = 0)$ may be obtained:

$$\omega L/c_2 = (\sqrt{3}/2)\beta U'_0; \tag{22a}$$

$$U_M/L = [1/(2\beta)]U'_0. \tag{22b}$$

At $x = L/2$, $U = U_M$, and $U'(x = L/2) = 0$, so, by equation (15), $|U''(x = L/2)| = \infty$. This last result may be made more precise by differentiating equation (16) with respect to x to get eventually

$$U''(x) = -\tfrac{1}{2}\sqrt{8/(3\sigma)}\,\omega U/(U_M^2 - U^2)^{1/2}. \tag{23}$$

This shows analytically how, as $U \to U_M - 0$, $U'' \to -\infty$ proportionately to $-1/(U_M - U)^{1/2}$.

2.2.2. *Approximate solution of the spatial equation*

The *Ansatz* for the fundamental mode of equation (15)

$$U(x) = a \sin(\pi x/L) \tag{24}$$

together with the first-order spatial equivalent of harmonic balance yields, in agreement with equations (11)–(13) above,

$$\Omega_1 = 3{\cdot}0219\ Y_M, \tag{25}$$

where $Y_M = U_M = a$ here.

This first-order approach (24) to solving the spatial equation (15) gives the approximate value (25) which is not in very good agreement with the exact value (19) for the spatial equation, being over 20% too high. In this case it is worthwhile to find a second approximation to the spatial equation (15) by using the second order equivalent of harmonic balance, i.e., a simplified Galerkin-type procedure. Thus the function used to describe $U(x)$ for the fundamental mode now also contains the third spatial harmonic (being symmetrical about its mid-point), and one sets

$$U(x) = a \sin(\pi x/L) + b \sin(3\pi x/L). \tag{26}$$

Now

$$U(x = L/2) = U_M = a + |b|, \tag{27}$$

where U_M is still the maximum, *provided* that $b < 0$; this must be checked *a posteriori*. It will also subsequently be checked that the obtained value of b/a is small and leads to an acceptable expression for a fundamental mode shape.

Retention of first and third spatial harmonics in equation (15) then gives

$$(L^4/\pi^4)\omega^2 = (3\sigma/16)a^2[1 + 3\alpha + 18\alpha^2], \tag{28}$$

where

$$\alpha = b/a \tag{29}$$

is the small solution of the algebraic equation

$$1 + 17\alpha - 3\alpha^2 + 63\alpha^3 = 0. \tag{30}$$

Thus

$$\alpha \approx -1/17 \approx -0{\cdot}059. \tag{31}$$

[Solution of the corresponding quadratic or cubic equation (30) gives $\alpha \approx -0{\cdot}058$ but the above fraction value (31) is sufficiently accurate for the present purposes.] This α is indeed negative, and so the coefficient b in equation (26) is negative and equation (27) holds. The coefficient (31) is sufficiently small so that $U'(x) > 0$, $0 \leqslant x < L/2$; so equation (26) can reasonably represent a fundamental mode shape.

The term in the square brackets in equation (28) for ω^2 then has a value of $16^2/17^2$; and from (27) $a = (17/18)U_M$. Thus ω is decreased by a factor of $(16/17)(17/18) = 8/9 = 0{\cdot}0889$, and

$$\Omega_2^S = 2{\cdot}686\ Y_M. \tag{32}$$

This reduces the harmonic balance error in (25) compared with (19) to about 8% which is more acceptable.

It is noted that the spatial differential equation (15) has a rather unusual form, not at all like a Helmholtz equation involving U'' and U. Thus it is not surprising that equation (26) does not lead to a highly accurate value for the frequency. The fact that a reasonable value is nevertheless obtained is perhaps a tribute to the harmonic balance type approach.

3. DISCUSSION

For the pretensionless string equation (3) under consideration, the approximate fundamental frequency solutions (13) and (32) to the temporal and spatial ordinary differential equations (6) and (15) respectively have been found. The former is very close to its exact value, and the latter is in reasonable agreement with its exact value. It is fortunate that the respective exact values (8) and (19) are both analytically available. They actually differ somewhat from each other. A closer reconciliation would require a more comprehensive consideration of non-linear normal modes of the original non-linear partial differential equation (3); this will not be pursued here.

In conclusion, it is noted that all the used solution approaches lead to $\Omega \propto Y_M$, and so direct comparisons are achievable by consideration of single numerical coefficients.

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